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GLOBAL ITERATION SCHEMES FOR MONOTONE OPERATORS.(U)

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GLOBAL ITERATION SCHEMES FOR MONOTONE OPERATORS

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ABSTRACT

We give three globally convergent iteration schemes for finding zeros of maximal monotone operators in Hilbert spaces. We assume that the operators are defined in the whole space and are either continuous, grow at most linearly at infinity or map bounded sets into bounded sets. As applications we have globally convergent iteration schemes for minimizing convex functionals in Hilbert spaces.

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SIGNIFICANCE AND EXPLANATION

One of the main problems in numerical mathematics is to construct solutions to equations $f(x) = 0$. It is important to know whether solutions can be found by iteration, i.e. using only a countable number of evaluations of f. In this paper we give convergent iteration schemes for operators which in a mathematical sense form a well defined general class, including, e.g., minimization of convex functionals. Although in many practical cases the convex functional would have some additional properties which imply that the usual gradient method would converge, we do not know of any previous iterative algorithms which would always converge for the general class of functionals considered here.

Our schemes all have the following important properties: only qualitative assumptions on the operators are made, one does not have to know whether solutions exist, and the schemes work independently of the initial guess. On the other hand, demanding all these properties implies that the convergence may be very slow.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

GLOBAL ITERATION SCHEMES FOR MONOTONE OPERATORS

Olavi Nevanlinna

1. Introduction.

In this paper we consider iteration schemes to find zeros of monotone operators in Hilbert spaces. Let H be a real Hilbert space. A possibly multivalued operator A is said to be monotone if

$$(1.1)\qquad (y_1 - y_2, x_1 - x_2) \geq 0 \quad \text{for all} \quad x_i \in D(A) \quad \text{and} \quad y_i \in Ax_i \quad .$$

A is maximal monotone if it is monotone and there does not exist any monotone proper extension of A. Examples of maximal monotone operators are subdifferentials $\partial\varphi$ of proper convex lower semicontinuous functionals φ, and finding solution of $\partial\varphi(x) \ni 0$ is equivalent to minimizing φ.

We shall show, using ideas of Bruck [3] and Crandall and Pazy [4], that if A is defined in the whole space and is either continuous or grows only linearly at infinity, then we can find sequences $\{\lambda_n\}$, $\{\theta_n\}$ such that if

$$(1.2)\qquad x_{n+1} \in x_n - \lambda_n(Ax_n + \theta_n x_n) \quad ,$$

then x_n converges strongly to a solution of $Ax \ni 0$, if there exists any, otherwise it tends to infinity. In particular, we can find the minimum of any lower semicontinuous convex functional which is either continuously differentiable or grows only quadratically at infinity.

Assume for a while that A is strongly monotone, i.e. it is of the form $bI + B$, where $b > 0$, I is the identity operator and B is maximal monotone. Then $R(bI + B) = H$ and $bx + Bx \ni 0$ has a unique solution. Any strictly contractive mapping $T:H \to H$, $|Tx - Ty| \leq k|x - y|$, $k < 1$, gives an example of such operators by defining $A = I - T$ $(= (1-k)I + (kI-T))$, and the zero of A, or the fixed point of T, can be obtained by Picard-Lindelöf iteration

$$(1.3)\qquad x_{n+1} = Tx_n \quad .$$

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If B is Lipschitz-continuous then we can still iterate in the form

$$x_{n+1} = x_n - \lambda(Bx_n + bx_n) \quad , \tag{1.4}$$

provided λ is small enough, and this was used by Zarantonello [8] to prove the existence of a zero for those operators. Lipschitz-continuity can be weakened, in a natural way, into two directions: assuming that the operator satisfies a linear growth condition or assuming only continuity. In both cases the zero can be found by iteration in the form

$$x_{n+1} \in x_n - \lambda_n(Bx_n + bx_n) \quad , \tag{1.5}$$

where $\{\lambda_n\}$ is a sequence of possitive numbers. Bruck [2] considered (1.5) with $\lambda_n = \frac{1}{cn+\sigma}$ for operators which have open domains, but his result is then local in the sense that convergence is guaranteed only when the initial value is chosen close enough to the solution. In section 4 we state a global version of this result by assuming that A satisfies a linear growth condition.

Continuous monotone operators are maximal and therefore $bx + Bx = 0$ has a solution. In [4] Crandall and Pazy gave a constructive proof for the existence of a solution in general Banach spaces. It uses iteration of the form (1.5) where λ_n is decided at each step using a finite number of evaluations of Ax.

If the operator is not strongly monotone schemes of the form

$$x_{n+1} \in x_n - \lambda_n A x_n \tag{1.6}$$

can still be used if the operator satisfies a convergence condition [6], but without some additional properties, even when A is the gradient of a convex functional, we generally do not have strong convergence. To obtain strong convergence in such a case we use the method of regularization: we apply the scheme (1.6) to operators $A + \theta_n I$ and let θ_n tend to zero as $n \to \infty$. The problem is to define $\{\lambda_n\}$ and $\{\theta_n\}$ in such a balanced way that strong convergence to a correct solution is obtained. Here we use ideas that go back to Bruck [3] and Halpern [5].

We also show that if A is defined in the whole space and is bounded, then we can always define a convergent scheme of the form (1.2), which uses a finite number of reinitializations.

2. Results.

Let A be a maximal monotone operator in a Hilbert space H. In our first result A is assumed to be continuous and in the second A satisfies a growth condition of the form

$$|y| \leq C(1 + |x|), \quad \text{for all } x, \text{ and } y \in Ax \quad . \tag{2.1}$$

Finally, in the third result we assume that A maps bounded sets into bounded sets. The algorithms can be applied without knowing whether there exists solutions or not, and we do not assume anything on the modulus of continuity, nor one has to know the value of the constant C in (2.1).

We state the result first for continuous operators. Let $\{h_n\}$ be a decreasing sequence of reals such that $\lim_{n\to\infty} h_n = 0$, and let $\{r_n\}$ be a decreasing sequence satisfying $\lim_{n\to\infty} r_n = 0$, $\lim_{n\to\infty} \frac{r_n}{r_{n+1}} = 1$. Choose a decreasing function $\psi(s)$ satisfying

$$\int_0^\infty \psi(s)ds < \infty \quad , \tag{2.2}$$

and such that for all $s_0 > 0$ there exists $\delta(t,s_0)$ so that

$$\psi(s) \leq \delta(t,s_0)\,\psi(s+t), \quad \text{for } s \geq s_0,\ t \geq 0 \quad . \tag{2.3}$$

For example, $\psi(s) = s^{-a}$ with $a > 1$ is such a function. Finally fix a number $\alpha > 0$. Set $\lambda_0 = h_0$, $\theta_0 = r_0$ and $n(0) = 0$, where $n(i)$ will be an increasing sequence of integers, such that $\theta_n = r_i$ for $n(i) \leq n < n(i+1)$. Now let λ_n be the largest $h \in \{h_n\}$ such that

$$|A(x_n - h(Ax_n + \theta_n x_n)) - Ax_n - h\theta_n(Ax_n + \theta_n x_n)| \leq \psi\Big(\sum_{j=0}^{n-1} \lambda_j\Big) \quad . \tag{2.4}$$

Such an h exists since A is continuous. If

$$\theta_n \sum_{j=n(i)}^{n} \lambda_j \geq \alpha \tag{2.5}$$

then set $n(i+1) = n + 1$, and $\theta_{n+1} = r_{i+1}$, otherwise keep $\theta_{n+1} = r_i$.

<u>Theorem 1</u>. Let A be a maximal monotone continuous operator in a Hilbert space with $D(A) = H$. Assume that $\{x_n\}$ satisfies (1.2) with any initial value $x_0 \in H$, where

$\{\lambda_n\}$, $\{\theta_n\}$ are defined recursively as above by (2.4) and (2.5). If $A^{-1}0 \neq \emptyset$, then x_n converges strongly to p, where p is the element in $A^{-1}0$ with minimum norm, and if $A^{-1}0 = \emptyset$, then $|x_n| \to \infty$ as $n \to \infty$ in such a way that $\theta_n x_n$ converges to $-a^0$ where a^0 is the element in $\overline{R(A)}$ with minimum norm.

□

The proof is postponed to section 3.

If instead of continuity we assume that A satisfies a growth condition of the form (2.1) then the sequences $\{\lambda_n\}$, $\{\theta_n\}$ can be chosen independently of the operator A. The following definition is essentially as in [3]:

Definition. Two sequences $\{\lambda_n\}$, $\{\theta_n\}$ of positive real numbers are acceptably paired if $\{\theta_n\}$ is nonincreasing, $\lim_{n\to\infty} \theta_n = 0$, and there exists an increasing sequence $\{n(i)\}$ of integers such that

(2.6) $$\liminf_{n\to\infty} \theta_{n(i)} \sum_{j=n(i)}^{n(i+1)-1} \lambda_j > 0$$

(2.7) $$\limsup_{n\to\infty} \theta_{n(i)} \sum_{j=n(i)}^{n(i+1)-1} \lambda_j < \infty$$

(2.8) $$\lim_{i\to\infty} (\theta_{n(i)} - \theta_{n(i+1)}) \sum_{j=n(i)}^{n(i+1)-1} \lambda_j = 0 \quad .$$

□

Part of the proof of Theorem 1 consists of showing that the recursive procedure (2.4), (2.5) always generates acceptably paired sequences. Another example of acceptably paired sequences is the following [3]: $\lambda_n = n^{-1}$, $\theta_n = (\log \log n)^{-1}$, $n(i) = i^i$.

Theorem 2. Let A be a maximal monotone operator in a Hilbert space and assume that it is defined in the whole space and satisfies the growth condition (2.1) for some $C < \infty$. Assume that $\{\lambda_n\} \in \ell^2$ and that $\{\lambda_n\}$, $\{\theta_n\}$ are acceptably paired. If $\{x_n\}$ satisfies (1.2) with any initial value $x_0 \in H$, then the conclusion of Theorem 1 holds.

□

The proof of this result is given in section 4.

Assume now that A is defined in the whole space and is only bounded, i.e. it maps bounded sets into bounded sets. We give an iteration scheme which uses reinitialization if

x_n tends to be too big.

Consider the scheme

$$x_{n+1}^{\mu} \in x_n^{\mu} - \gamma_{\mu}\lambda_n(Ax_n^{\mu} + \theta_n x_n^{\mu}), \quad x_0^{\mu} = x_0 \quad , \tag{2.9}$$

where γ_{μ} is any decreasing sequence such that $\lim_{\mu\to\infty} \gamma_{\mu} = 0$, and $\{\lambda_n\}$, $\{\theta_n\}$ are acceptably paired and $\{\lambda_n\} \in \ell^2$. Choose an increasing sequence $\{R_{\mu}\}$ such that $\lim_{\mu\to\infty} R_{\mu} = \infty$. Now compute $\{x_n^{\mu}\}$ for a fixed μ using (2.9), as long as $|x_n^{\mu} - x_0| \leq R_{\mu}$. If $|x_{n+1}^{\mu} - x_0| > R_{\mu}$, then start computing the sequence $\{x_n^{\mu+1}\}$.

<u>Theorem 3.</u> Let A be a bounded maximal monotone operator, and assume that $\{x_n^{\mu}\}$ is obtained using the procedure described above. If $A^{-1}0 \neq \emptyset$, then, for some μ, $|x_n^{\mu} - x_0| \leq R_{\mu}$ for all n, and x_n^{μ} tends to the minimum element in $A^{-1}0$; otherwise $A^{-1}0 = \emptyset$ and μ tends to infinity.

$\square$

<u>Proof</u>. We observe first that if $A^{-1}0 =$ [illegible] μ tends to infinity. In fact, otherwise for some fixed μ, $\{x_n^{\mu}\} \subset B_{R_{\mu}}(x_0)$ for all n, where $B_s(z)$ denotes the closed ball of radius s, centered at z. But then A, operating only in $B_{R_{\mu}}(x_0)$ can be thought of as satisfying the growth condition (2.1) and, by Theorem 2, we would have $A^{-1}0 \neq \emptyset$ and x_n^{μ} converging to the minimum element in $A^{-1}0$. Therefore, the only thing we have to prove is, that if $A^{-1}0 \neq \emptyset$ then μ cannot tend to infinity.

Consider the scheme

$$x_{n+1} \in x_n - \gamma\lambda_n(Ax_n + \theta_n x_n) \quad , \tag{2.10}$$

where $\gamma > 0$ and $\{\lambda_n\}$, $\{\theta_n\}$ are as in (2.9). A theorem of Bruck [3, Theorem 3] includes the following result: If $y \in A^{-1}0$ and for some $r \geq |y|$, $x_0 \in B_r(y)$ and A is defined and bounded in $B_{2r}(y)$, then there exists a $\gamma^* > 0$ such that if $\gamma = \gamma^*$, then x_n stays in $B_{2r}(y)$ and converges to the minimum element in $A^{-1}0$. It is evident from the proof of this result that the same conclusion actually holds for all $\gamma \in (0,\gamma^*]$.

If $A^{-1}0 \neq \emptyset$, let $r = \max\{|y|, |x_0 - y|\}$, for some $y \in A^{-1}0$. Then, as μ grows, there is a μ_0 such that $\gamma_{\mu_0} \leq \gamma^*$ and $R_{\mu_0} \geq 3r$, so that we must have

$\{x_n^{\mu_{max}}\} \subset B_{2r}(y) \subset B_{R_{\mu_0}}(x_0)$ for some $\mu_{max} \leq \mu_0$, and the proof is complete.

□

Continuous monotone operators are maximal, but in the situations discussed by Theorems 2 and 3 it may happen that we do not know the maximality of the operator. However, even if the operator is not maximal, the schemes are well defined and the conclusions hold, with the modification that the convergence takes place exactly when there exists a monotone extension $\tilde{A}$ of the operator A such that $\tilde{A}^{-1}0 \neq \emptyset$.

3. Proof of Theorem 1.

The proof consists formally of two parts. Denote by p_n the unique vector such that

$$(3.1) \qquad \theta_n p_n + Ap_n \ni 0 \quad .$$

We show that the sequence $\{x_n\}$ generated by (1.2), (2.4), (2.5) behaves for large n as $\{p_n\}$, and, in fact this is true in any Banach space for continuous accretive operators. However, in general $A^{-1}0 \neq \emptyset$ does not imply that $\{p_n\}$ converges. In Hilbert spaces this always happens and in particular the convergence of $\{x_n\}$ follows.

Let $J_\lambda = (I + \lambda A)^{-1}$, so that $p_n = J_{\frac{1}{\theta_n}}0$. Now the behavior of p_n follows from the following result.

Lemma. Let A be any maximal monotone operator in a Hilbert space. Then

$$(3.2) \qquad \lim_{\lambda \to \infty} \frac{1}{\lambda} J_\lambda 0 = -a^0 \quad ,$$

where a^0 is the element in $\overline{R(A)}$ with minimum norm. If $A^{-1}0 = \emptyset$ then $|J_\lambda 0| \to \infty$ as $\lambda \to \infty$, and if $A^{-1}0 \neq \emptyset$, then $J_\lambda 0 \to p$ as $\lambda \to \infty$ where p is the element in $A^{-1}0$ with minimum norm.

□

Proof. Let B_t denote the Yosida approximation of a maximal monotone operator B: $B_t = t^{-1}(I - J_t(B))$, where $J_t(B) = (1 + tB)^{-1}$. One verifies easily that $J_t(B)0 = (B^{-1})_{\frac{1}{t}} 0$. Now $J_t(B)0$ tends to the minimum element in $\overline{D(B)}$ as $t \to 0$ [1, Theorem 2.2]. Therefore $(B^{-1})_{\frac{1}{t}} 0$ tends to the minimum element in $\overline{D(B)}$ as $t \to 0$. For $B^{-1} = A$ this means that $\lim_{t\to\infty} A_t 0 = a^0$, and (3.2) follows since $\frac{1}{t} J_t 0 = -A_t 0$. We also know [1, Proposition 2.6] that $B_t 0$ tends to the minimum element in $B0$ as $t \to 0$ if $0 \in D(B)$, otherwise $|B_t 0| \to \infty$. Using this for $(A^{-1})_{\frac{1}{t}} 0$ with $t \to \infty$ yields the rest of the Lemma.

□

We consider first the following implicit iteration scheme

$$(3.3) \qquad x_{n+1} + \lambda_n(Ax_{n+1} + \theta_n x_{n+1}) \ni x_n + e_n \quad .$$

<u>Proposition 1</u>. Let A be maximal monotone and $\{\lambda_n\}$, $\{\theta_n\}$ be acceptably paired. Assume that $\{x_n\}$ satisfies (3.3) so that $\sum_0^\infty |e_n| < \infty$. If $A^{-1}0 \neq \emptyset$, then x_n converges to the minimum element in $A^{-1}0$, otherwise $|x_n| \to \infty$ in such a way that $-\theta_n x_n$ converges to the minimum element in $\overline{R(A)}$.

□

<u>Proof</u>. There exists a unique sequence $\{y_n\}$ with $y_n \in Ax_n$ such that (3.3) holds. For simplicity we shall denote the element also by Ax_n. First we reduce the problem to the case where $e_n \equiv 0$. In fact, if the conclusion holds for $e_n \equiv 0$ it also holds if $\{e_n\}$ has compact support. Then, approximating $\{\tilde{e}_n\} \in \ell^1$ by a sequence $\{e_n\}$ with a compact support and using the fact that the resolvents are contractions we have

$$|x_n - \tilde{x}_n| \le \varepsilon \tag{3.4}$$

if $\sum_{j=0}^\infty |e_j - \tilde{e}_j| < \varepsilon$, which completes the reduction.

Subtract $p_{n(i)}$ from both sides of (3.3) to get

$$(1 + \lambda_n\theta_n)x_{n+1} - p_{n(i)} + \lambda_n Ax_{n+1} = x_n - p_{n(i)} \ . \tag{3.5}$$

Using (3.1) we obtain

$$\begin{aligned} &(1 + \lambda_n\theta_n)(x_{n+1} - p_{n(i)}) + (\lambda_n\theta_n - \lambda_n\theta_{n(i)})p_{n(i)} \\ &\quad + \lambda_n Ax_n - \lambda_n Ap_{n(i)} = x_n - p_{n(i)} \ , \end{aligned} \tag{3.6}$$

where $Ap_{n(i)}$ stands for the element satisfying $\theta_{n(i)}p_{n(i)} + Ap_{n(i)} \ni 0$. Since A is monotone (3.6) yields

$$(1 + \lambda_n\theta_n)|x_{n+1} - p_{n(i)}| - \lambda_n|\theta_n - \theta_{n(i)}|\ |p_{n(i)}| \le |x_n - p_{n(i)}|$$

and iterating this from $n(i)$ on we obtain

$$|x_n - p_{n(i)}| \le \prod_{j=n(i)}^{n-1} (1 + \lambda_j\theta_j)^{-1}|x_{n(i)} - p_{n(i)}| + \sum_{j=n(i)}^{n-1} \lambda_j|\theta_{n(i)} - \theta_j|\ |p_{n(i)}| \ . \tag{3.7}$$

Since θ_j is nonincreasing we have

$$\prod_{j=n(i)}^{n(i+1)-1} (1 + \lambda_j \theta_j)^{-1} \le \prod_{j=n(i)}^{n(i+1)-1} (1 + \lambda_j \theta_{n(i+1)})^{-1} .$$

But (2.6) and (2.8) imply that there exists $\alpha > 0$ such that $\theta_{n(i+1)} \sum_{j=n(i)}^{n(i+1)-1} \lambda_j \ge \alpha$ and hence for some $\delta < 1$, $\prod_{j=n(i)}^{n(i+1)-1} (1 + \lambda_j \theta_j)^{-1} \le \delta$. For the second term in (3.7) we obtain

$$\varepsilon_{n(i)} \stackrel{\text{def}}{=} \sum_{j=n(i)}^{n(i+1)-1} \lambda_j |\theta_{n(i)} - \theta_j| \le \sum_{j=n(i)}^{n(i+1)-1} \lambda_j |\theta_{n(i)} - \theta_{n(i+1)}| \to 0 \quad \text{as} \quad i \to \infty \quad \text{by (2.8).}$$

Hence, for $n(i) \le n \le n(i+1)$ we have

$$(3.8) \qquad |x_n - p_{n(i)}| \le |x_{n(i)} - p_{n(i)}| + \varepsilon_{n(i)} |p_{n(i)}|$$

and in particular

$$(3.9) \qquad |x_{n(i+1)} - p_{n(i)}| \le \delta |x_{n(i)} - p_{n(i)}| + \varepsilon_{n(i)} |p_{n(i)}| .$$

Since A is monotone we have

$$(3.10) \qquad \begin{aligned} |p_{n(i)} - p_{n(i+1)}| &\le |p_{n(i)} - p_{n(i+1)} + \frac{1}{\theta_{n(i+1)}} [Ap_{n(i)} - Ap_{n(i+1)}]| \\ &= ([\theta_{n(i)} - \theta_{n(i+1)}] / \theta_{n(i+1)}) |p_{n(i)}| . \end{aligned}$$

From (2.6)-(2.8) we see that $\lim_{i\to\infty} \left(\frac{\theta_{n(i)}}{\theta_{n(i+1)}} - 1\right) = 0$. Hence (3.9) and (3.10) yield

$$(3.11) \qquad \begin{aligned} |x_{n(i+1)} - p_{n(i+1)}| &\le |x_{n(i+1)} - p_{n(i)}| + |p_{n(i)} - p_{n(i+1)}| \\ &\le \delta |x_{n(i)} - p_{n(i)}| + \alpha_{n(i)} |p_{n(i)}| , \end{aligned}$$

where $\alpha_{n(i)} = \varepsilon_{n(i)} + \frac{\theta_{n(i)}}{\theta_{n(i+1)}} - 1 \to 0$ as $i \to \infty$. But (3.10) and (3.11) imply that

$$(1 - \alpha_{n(i)}) \frac{|x_{n(i+1)} - p_{n(i+1)}|}{1 + |p_{n(i+1)}|} \le \delta \frac{|x_{n(i)} - p_{n(i)}|}{1 + |p_{n(i)}|} + \alpha_{n(i)}$$

holds for $\alpha_{n(i)} \to 0$ and $\delta < 1$ and therefore

(3.12) $$\lim_{i\to\infty} \frac{|x_{n(i)} - p_{n(i)}|}{1 + |p_{n(i)}|} = 0 \quad .$$

Using (3.8) we then obtain

(3.13) $$\lim_{i\to\infty} \max_{n(i)\le n\le (i+1)} \frac{|x_n - p_{n(i)}|}{1 + |p_{n(i)}|} = 0 \quad .$$

If $A^{-1}0 \neq \emptyset$ then by Lemma , $p_{n(i)} \to p$, where p is the minimum element in $A^{-1}0$, and by (3.13), we also have $\lim_{n\to\infty} |x_n - p| = 0$. Assume then that $A^{-1}0 = \emptyset$. By Lemma, $|p_n| \to \infty$ and $-\theta_n p_n \to a^0$. From (3.13) it follows immediately that $|x_n| \to \infty$ as $n \to \infty$. To prove $\theta_n x_n \to -a^0$, consider, for $n(i) \le n \le n(i+1)$,

(3.14) $$\frac{|\theta_n x_n - \theta_{n(i)} p_{n(i)}|}{\theta_{n(i)} + \theta_{n(i)} |p_{n(i)}|} \le \frac{|x_n - p_{n(i)}|}{1 + |p_{n(i)}|} + \frac{|\theta_{n(i)} - \theta_n|}{\theta_{n(i)}} \frac{|x_n|}{1 + |p_{n(i)}|} \quad .$$

The first term on the right in (3.14) tends to zero as $i \to \infty$ *by (3.13), and in the second* term $\frac{|x_n|}{1 + |p_{n(i)}|}$ stays bounded while $\frac{|\theta_{n(i)} - \theta_n|}{\theta_{n(i)}} \to 0$ as $i \to \infty$. Since $\theta_{n(i)} p_{n(i)} \to -a^0$, we finally obtain $\theta_n x_n \to -a^0$ as $n \to \infty$.

□

Schemes of the form (3.3) have also been discussed in [7].

Assume now that A is defined in the whole space and is continuous. Following the idea in [4] we write the explicit scheme (1.2) as an implicit scheme (3.3) with errors:

(3.15) $$x_{n+1} + \lambda_n (Ax_{n+1} + \theta_n x_{n+1}) = x_n + e_n \quad ,$$

where

(3.16) $$e_n = \lambda_n (Ax_{n+1} - Ax_n + \theta_n (x_{n+1} - x_n)) \quad .$$

__Proposition 2.__ If $\{\lambda_n\}$ and $\{\theta_n\}$ are chosen according to the procedure described by (2.4) and (2.5), then $\sum_{n=0}^{\infty} |e_n| < \infty$, and $\{\lambda_n\}$, $\{\theta_n\}$ are acceptably paired.

□

__Proof.__ Using (2.4) we have

$$\sum_{n=0}^{\infty} |e_n| = \sum_{n=0}^{\infty} \lambda_n |Ax_{n+1} - Ax_n + \theta_n (x_{n+1} - x_n)|$$

$$\leq \sum_{n=0}^{\infty} \lambda_n \, \psi\Big(\sum_{j=0}^{n-1} \lambda_j\Big) \quad .$$

But (2.2) and (2.3) imply that

$$\sum_{n=0}^{\infty} \lambda_n \, \psi\Big(\sum_{j=0}^{n-1} \lambda_j\Big) \leq \delta(h_0, h_0) \sum_{n=0}^{\infty} \lambda_n \, \psi\Big(\sum_{j=0}^{n} \lambda_j\Big)$$

$$\leq \delta(h_0, h_0) \int_0^{\infty} \psi(s)\, ds < \infty \quad ,$$

which proves that $\sum_{n=0}^{\infty} |e_n| < \infty$.

If (2.5) holds, then we have

$$\alpha + r_0 h_0 \geq \theta_{n(i)} \sum_{j=n(i)}^{n(i+1)-1} \lambda_j \geq \alpha$$

since $\lambda_j \leq h_0$ and $\theta_{n(i)} \leq r_0$ and together with $r_n / r_{n+1} \to 1$ these imply that $\{\lambda_n\}$, $\{\theta_n\}$ are acceptably paired. Assume therefore that for some $n(i)$ there does not exist any n such that (2.5) holds, i.e.

(3.17) $$\theta_{n(i)} \sum_{j=n(i)}^{n} \lambda_j < \alpha \quad \text{for all } n \quad .$$

This leads to a contradiction: Consider the scheme (3.3) with $\theta_n \equiv \theta_{n(i)} \equiv \theta > 0$ and assume that $\sum_{n=0}^{\infty} \lambda_n < \infty$ and $\sum_{n=0}^{\infty} |e_n| < \infty$. One proves easily that x_n converges to some x_∞. In fact, if $\{e_n\}$ has compact support then, for large enough n, $|Ax_n + \theta x_n|$ is nonincreasing and therefore for some $C < \infty$

$$\sum_{n=0}^{\infty} |x_{n+1} - x_n| \leq C \sum_{n=0}^{\infty} \lambda_n + \sum_{n=0}^{\infty} |e_n| \quad ,$$

and so $\{x_n\}$ is a Cauchy sequence. The general case follows using the same approximation argument as in the proof of Proposition 1.

Since A is continuous there is an $h \in \{h_n\}$ such that

$$|A(x_\infty - h(Ax_\infty + \theta x_\infty)) - Ax_\infty - h\theta(Ax_\infty + \theta x_\infty)|$$

$$\leq \frac{1}{2}\psi(\sum_{j=0}^{\infty}\lambda_j) \quad .$$

But then also

$$|A(x_n - h(Ax_n + \theta x_n)) - Ax_n - h\theta(Ax_n + \theta x_n)|$$

$$\leq \psi(\sum_{j=0}^{n-1}\lambda_j)$$

for large enough n and therefore $\lambda_n \geq h$, contradicting the assumption $\{\lambda_n\} \in \ell^1$.

□

Theorem 1 follows by combining Proposition 1 and Proposition 2.

□

4. Proof of Theorem 2.

In [3] Bruck proves a convergence result for general maximal monotone operators using (1.2), under the assumption that $A^{-1}0 \neq \emptyset$, $\{x_n\}$ is defined, and both $\{x_n\}$, $\{y_n\}$, $y_n \in Ax_n$ are bounded. He then applies the result to prove the existence of a convergent iteration scheme for operators which are bounded in a ball centered by a $p \in A^{-1}0$. The scheme is defined, however, only after we have a priori knowledge on the location of $A^{-1}0$ and the boundedness of A. Theorem 2 is a different version of Bruck's result in the sense that it only assumes a qualitative growth condition to be satisfied; one does not have to know that solutions exist and one can start from any initial vector.

We could prove the part of Theorem 2 where $A^{-1}0 \neq \emptyset$ by showing that $\{x_n\}$ is bounded and then using the result of Bruck [3]. However, when showing that $\{x_n\}$ is bounded we essentially shall get the convergence as well, and therefore we give here the complete proof.

Let again p_n be defined by $\theta_n p_n + Ap_n \ni 0$. We also denote by Ap_n the element w_n in the set Ap_n which satisfies $\theta_n p_n + w_n = 0$, and, similarly, we denote by Ax_n the vector $(x_n - x_{n+1})/\lambda_n - \theta_n x_n$. Subtracting $p_{n(i)}$ from both sides of (1.2) yields

$$x_{n+1} - p_{n(i)} = x_n - p_{n(i)} - \lambda_n(Ax_n + \theta_n x_n) \quad . \tag{4.1}$$

Squaring both sides of (4.1) and rearranging yields

$$\begin{aligned} |x_{n+1} - p_{n(i)}|^2 = |x_n - p_{n(i)}|^2 &+ 2\lambda_n(\theta_{n(i)} - \theta_n)(x_n, x_n - p_{n(i)}) \\ &- 2\lambda_n(Ax_n + \theta_{n(i)}x_n, x_n - p_{n(i)}) \\ &+ \lambda_n^2|Ax_n + \theta_n x_n|^2 \quad . \end{aligned} \tag{4.2}$$

Since A is monotone and $\theta_{n(i)}p_{n(i)} + Ap_{n(i)} = 0$, we have $(Ax_n + \theta_{n(i)}x_n, x_n - p_{n(i)}) \geq \theta_{n(i)}|x_n - p_{n(i)}|^2$. Using (2.1) the last term on the right of (4.2) can be bounded in the form

$$|Ax_n + \theta_n x_n|^2 \leq C_1\{1 + |x_n - p_{n(i)}|^2 + |p_{n(i)}|^2\} \quad .$$

For the second term we write

$$|(x_n, x_n - p_{n(i)})| \leq |x_n - p_{n(i)}|^2 + \frac{\eta}{2}|x_n - p_{n(i)}|^2 + \frac{1}{2\eta}|p_{n(i)}|^2 \quad .$$

Substituting these into (4.2) yields

$$|x_{n+1} - p_{n(i)}|^2 \le [1 - \beta_{n,n(i)}]|x_n - p_{n(i)}|^2 + \gamma_{n,n(i)}(1 + |p_{n(i)}|^2) \tag{4.3}$$

where

$$\beta_{n,n(i)} = 2\lambda_{n-1}\theta_{n(i)} - 2(1 + {}^{\eta}/2)\lambda_n|\theta_{n(i)} - \theta_n| - C_1\lambda_n^2 \quad ,$$

and

$$\gamma_{n,n(i)} = ({}^{\lambda}n/\eta)|\theta_{n(i)} - \theta_n| + C_1\lambda_n^2 \quad .$$

Since $\{\lambda_n\} \in \ell^2$, and the sequences are acceptably paired we have for large enough $n(i)$ and some $\alpha_1 > 0$, $\alpha_2 < \infty$ that

$$\sum_{j=n(i)}^{n(i+1)-1} \beta_{j,n(i)} \ge \alpha_1 \quad \text{and} \quad \sum_{j=n(i)}^{n} \beta_{j,n(i)} \ge -\alpha_2 \tag{4.4}$$

for $n(i) \le n \le n(i+1)$, and if we denote

$$\sum_{j=n(i)}^{n(i+1)-1} \gamma_{j,n(i)} = \varepsilon_{n(i)} \quad ,$$

then $\lim\limits_{i\to\infty} \varepsilon_{n(i)} = 0$.

Iterating (4.3) from $n(i)$ on yields together with (4.4) that for $n(i)$ large enough we have

$$|x_{n(i+1)} - p_{n(i)}|^2 \le \delta|x_{n(i)} - p_{n(i)}|^2 + \varepsilon_{n(i)}\{1 + |p_{n(i)}|^2\} \tag{4.5}$$

for some $\delta < 1$, and for some $D < \infty$ and $n(i) \le n \le n(i+1)$,

$$|x_n - p_{n(i)}|^2 \le D|x_{n(i)} - p_{n(i)}|^2 + \varepsilon_{n(i)}\{1 + |p_{n(i)}|^2\} \quad . \tag{4.6}$$

Now one completes the proof as in the proof of Proposition 1 using (4.5) and (4.6) in place of (3.9) and (3.8).

□

To the end we state a simple result for operators of the form $bI + B$.

Proposition 3. Let B be maximal monotone, $D(B) = H$ and assume that B satisfies the growth condition (2.1). Assume also that $b > 0$ and $\{\lambda_n\} \in \ell^2 \backslash \ell^1$. If $\{x_n\}$ satisfies (1.5), then it converges to the unique solution of $bx + Bx \quad 0$.

$\square$

The proof is easier than that of Theorem 2 and is left to the reader. Related results are proved in [2], [6].

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We give three globally convergent iteration schemes for finding zeros of maximal monotone operators in Hilbert spaces. We assume that the operators are defined in the whole space and are either continuous, grow at most linearly at infinity or map bounded sets into bounded sets. ~~As applications we have~~ globally convergent iteration schemes for minimizing convex functionals in Hilbert spaces, are given as applications.

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